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NONLINEAR CONSTITUTIVE EQUATIONS FOR GRAVITOELECTROMAGNETISM

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This paper studies nonlinear constitutive equations for gravitoelectromagnetism. Eventually, the problem is solved of finding, for a given particular solution of the gravity-Maxwell equations, the exact form of the corresponding nonlinear constitutive equations.

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1. Introduction

Over the past decade, a description of nonlinear classical electrodynamics and Yang–Mills theory has been considered in the literature [1–3], with the hope of being able to extend it to a broader framework, including gauge theories of gravity [4] and quantum gravity [5].

However, no explicit calculation had been performed, and the formulation remained too general for the physics community to be able to appreciate its potentialities. For this purpose, as a first step, we here consider the gravitoelectromagnetism in the weak-field approximation (following, e.g. [6]). Recall the standard Maxwell equations in SI units [7]

$$\begin{aligned} \operatorname{curl}\mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \qquad \operatorname{div}\mathbf{B} = 0, \\ \operatorname{curl}\mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}, \quad \operatorname{div}\mathbf{D} = \rho, \end{aligned} \tag{1.1}$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, ρ is charge density, $\mathbf{j}$ is electric current density. In the linear case

$$\mathbf{B} = \mu_0 \mathbf{H}, \quad \mathbf{D} = \varepsilon_0 \mathbf{E}. \tag{1.2}$$

In the nonlinear case these equations can be presented in the form [8]

$$\begin{aligned} \mathbf{D} &= M(I_1, I_2)\mathbf{B} + \frac{1}{c^2} N(I_1, I_2)\mathbf{E}, \\ \mathbf{H} &= N(I_1, I_2)\mathbf{B} - M(I_1, I_2)\mathbf{E}, \end{aligned} \tag{1.3}$$

where the invariants are ($F_{\mu\nu}$ being the electromagnetic field tensor, with Hodge dual ${}^*F^{\mu\nu}$)

$$I_1 = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} = \mathbf{B}^2 - \frac{1}{c^2}\mathbf{E}^2, \quad I_2 = -\frac{c}{4} F_{\mu\nu} {}^*F^{\mu\nu} = \mathbf{B}\cdot\mathbf{E}. \tag{1.4}$$

Their gravitational analogues in SI are

$$\operatorname{curl}\mathbf{E}_g = -\frac{\partial \mathbf{B}_g}{\partial t}, \qquad \operatorname{div}\mathbf{B}_g = 0, \tag{1.5}$$

$$\operatorname{curl}\mathbf{B}_g = \frac{1}{c^2}\frac{\partial \mathbf{E}_g}{\partial t} + \frac{1}{\varepsilon_g c^2}\mathbf{j}_g, \quad \operatorname{div}\mathbf{E}_g = \frac{1}{\varepsilon_g}\rho_g, \tag{1.6}$$

where $\mathbf{E}_g$ is the static gravitational field (conventional gravity, also called gravitoelectric for the sake of analogy), $\mathbf{B}_g$ is the gravitomagnetic field, ρ_g is mass density, $\mathbf{j}_g$ is mass current density, G is the gravitational constant, ε_g is the gravity permittivity (analog of ε_0). Here

$$\varepsilon_g = -\frac{1}{4\pi G}, \quad \mu_g = -\frac{4\pi G}{c^2}, \tag{1.7}$$

are the gravitational permittivity and permeability, respectively.

The main idea is to introduce analogues of $\mathbf{H}$ and $\mathbf{D}$ to write (1.5) and (1.6) in the Maxwell form for four fields in SI as

$$\operatorname{curl}\mathbf{E}_g = -\frac{\partial \mathbf{B}_g}{\partial t}, \qquad \operatorname{div}\mathbf{B}_g = 0, \tag{1.8}$$

$$\operatorname{curl}\mathbf{H}_g = \frac{\partial \mathbf{D}_g}{\partial t} + \mathbf{j}_g, \quad \operatorname{div}\mathbf{D}_g = \rho_g. \tag{1.9}$$

In the linear-gravity case

$$\mathbf{D}_g = \varepsilon_g \mathbf{E}_g, \tag{1.10}$$

$$\mathbf{B}_g = \mu_g \mathbf{H}_g, \tag{1.11}$$

$$\varepsilon_g \mu_g = \frac{1}{c^2}. \tag{1.12}$$

Note now that the linear-gravity case (1.10)–(1.12) corresponds to weak approximation and some special case of gravitational field configuration. We generalize it to nonlinear case which can describe other configurations and non-weak fields, as in (1.3), by

$$\mathbf{D}_g = M_g(I_{g1}, I_{g2})\mathbf{B}_g + \frac{1}{c^2} N_g(I_{g1}, I_{g2})\mathbf{E}_g, \tag{1.13}$$

$$\mathbf{H}_g = N_g(I_{g1}, I_{g2})\mathbf{B}_g - M_g(I_{g1}, I_{g2})\mathbf{E}_g, \tag{1.14}$$

where the invariants are

$$I_{g1} = \mathbf{B}_g^2 - \frac{1}{c^2}\mathbf{E}_g^2, \quad I_{g2} = \mathbf{B}_g \cdot \mathbf{E}_g. \tag{1.15}$$

The gravity-Maxwell equations (1.8) and (1.9) together with the nonlinear gravity-constitutive equations (1.13) and (1.14) can give a nonlinear electrodynamics formulation of gravity (or at least some particular instances of this construction).

2. Linear Gravitoelectromagnetic Waves

The gravity-Maxwell equations for gravitoelectromagnetic waves (far from sources) are

$$\operatorname{curl} \mathbf{E}_g = -\frac{\partial \mathbf{B}_g}{\partial t}, \quad \operatorname{div} \mathbf{B}_g = 0, \tag{2.1}$$

$$\operatorname{curl} \mathbf{H}_g = \frac{\partial \mathbf{D}_g}{\partial t}, \quad \operatorname{div} \mathbf{D}_g = 0, \tag{2.2}$$

with generic values of permittivity and permeability (1.7). Then

$$\begin{aligned} \operatorname{curl} \mathbf{E}_g &= -\mu_g \frac{\partial \mathbf{H}_g}{\partial t}, \quad \operatorname{div} \mathbf{H}_g = 0, \\ \operatorname{curl} \mathbf{H}_g &= \varepsilon_g \frac{\partial \mathbf{E}_g}{\partial t}, \quad \operatorname{div} \mathbf{E}_g = 0. \end{aligned} \tag{2.3}$$

We differentiate the first equation with respect to time: $\operatorname{curl} \frac{\partial}{\partial t}\mathbf{E}_g = -\mu_g \frac{\partial^2 \mathbf{H}_g}{\partial t^2} \Rightarrow \frac{1}{\varepsilon_g}\operatorname{curl}(\operatorname{curl} \mathbf{H}_g) = -\mu_g \frac{\partial^2 \mathbf{H}_g}{\partial t^2}$. Since $\operatorname{curl}(\operatorname{curl} \mathbf{H}_g) = \operatorname{grad}(\operatorname{div} \mathbf{H}_g) - \Delta \mathbf{H}_g = -\Delta \mathbf{H}_g$, then

$$\Delta \mathbf{H}_g = \varepsilon_g \mu_g \frac{\partial^2 \mathbf{H}_g}{\partial t^2}. \tag{2.4}$$

By analogy, from the second equation $\operatorname{curl}\frac{\partial}{\partial t}\mathbf{H}_g = \varepsilon_g \frac{\partial^2 \mathbf{E}_g}{\partial t^2} \Rightarrow \frac{-1}{\mu_g}\operatorname{curl}(\operatorname{curl}\mathbf{E}_g) = \varepsilon_g \frac{\partial^2 \mathbf{E}_g}{\partial t^2}$. Hence we get the wave equation for $\mathbf{E}_g$,

$$\Delta \mathbf{E}_g = \varepsilon_g \mu_g \frac{\partial^2 \mathbf{E}_g}{\partial t^2}. \tag{2.5}$$

3. Nonlinear Gravitoelectromagnetic Waves

The differences begin with the constitutive equations (1.13) and (1.14). For simplicity put first $M_g = 0$. Then

$$\mathbf{D}_g = \frac{N}{c^2}\mathbf{E}_g, \tag{3.1}$$

$$\mathbf{B}_g = \frac{1}{N}\mathbf{H}_g, \tag{3.2}$$

where $N \equiv N_g(I_{g1}, I_{g2})$. The Maxwell equations become (hereafter the dots denote time derivatives)

$$\operatorname{curl}\mathbf{E}_g = -\left(\frac{1}{N}\right)^{\cdot}\mathbf{H}_g - \frac{1}{N}\frac{\partial \mathbf{H}_g}{\partial t}, \tag{3.3}$$

$$\operatorname{div}\left(\frac{1}{N}\mathbf{H}_g\right) = \mathbf{H}_g \operatorname{grad}\left(\frac{1}{N}\right) + \frac{1}{N}\operatorname{div}(\mathbf{H}_g) = 0, \tag{3.4}$$

$$\operatorname{curl}\mathbf{H}_g = \frac{\dot{N}}{c^2}\mathbf{E}_g + \frac{N}{c^2}\frac{\partial \mathbf{E}_g}{\partial t}, \tag{3.5}$$

$$\operatorname{div}\left(\frac{N}{c^2}\mathbf{E}_g\right) = \mathbf{E}_g \operatorname{grad}\left(\frac{N}{c^2}\right) + \frac{N}{c^2}\operatorname{div}(\mathbf{E}_g) = 0. \tag{3.6}$$

Take derivative of (3.3) with respect to time and get

$$\operatorname{curl}\frac{\partial}{\partial t}\mathbf{E}_g = -\left(\frac{1}{N}\right)^{\cdot\cdot}\mathbf{H}_g - 2\left(\frac{1}{N}\right)^{\cdot}\frac{\partial \mathbf{H}_g}{\partial t} - \frac{1}{N}\frac{\partial^2 \mathbf{H}_g}{\partial t^2}. \tag{3.7}$$

From (3.5), it follows $\frac{\partial \mathbf{E}_g}{\partial t} = \frac{c^2}{N}\operatorname{curl}\mathbf{H}_g - \frac{\dot{N}}{N}\mathbf{E}_g$. Then we get

$$\operatorname{curl}\left(\frac{c^2}{N}\operatorname{curl}\mathbf{H}_g - \frac{\dot{N}}{N}\mathbf{E}_g\right) = -\left(\frac{1}{N}\right)^{\cdot\cdot}\mathbf{H}_g - 2\left(\frac{1}{N}\right)^{\cdot}\frac{\partial \mathbf{H}_g}{\partial t} - \frac{1}{N}\frac{\partial^2 \mathbf{H}_g}{\partial t^2}. \tag{3.8}$$

The left-hand side here is

$$\begin{aligned}\operatorname{curl}\left(\frac{c^2}{N}\operatorname{curl}\mathbf{H}_g - \frac{\dot{N}}{N}\mathbf{E}_g\right) &= \operatorname{grad}\frac{c^2}{N} \times \operatorname{curl}\mathbf{H}_g + \frac{c^2}{N}\operatorname{grad}\operatorname{div}\mathbf{H}_g - \frac{c^2}{N}\Delta\mathbf{H}_g \\ &\quad - \frac{\dot{N}}{N}\operatorname{curl}\mathbf{E}_g - \operatorname{grad}\frac{\dot{N}}{N} \times \mathbf{E}_g.\end{aligned} \tag{3.9}$$

From (3.4), we get $\operatorname{div}(\mathbf{H}_g) = -N\mathbf{H}_g \operatorname{grad}(\frac{1}{N}) \neq 0$. Thus, the nonlinear analogue of the wave equation is

$$\operatorname{grad}\frac{c^2}{N} \times \operatorname{curl}\mathbf{H}_g + \frac{c^2}{N}\operatorname{grad}\operatorname{div}\mathbf{H}_g - \frac{c^2}{N}\Delta\mathbf{H}_g - \frac{\dot{N}}{N}\operatorname{curl}\mathbf{E}_g - \operatorname{grad}\frac{\dot{N}}{N} \times \mathbf{E}_g$$
$$= -\left(\frac{1}{N}\right)^{\cdot\cdot}\mathbf{H}_g - 2\left(\frac{1}{N}\right)^{\cdot}\frac{\partial \mathbf{H}_g}{\partial t} - \frac{1}{N}\frac{\partial^2 \mathbf{H}_g}{\partial t^2}. \tag{3.10}$$

Note that if $N = \text{const.}$, then we obtain the usual wave equation

$$\Delta\mathbf{H}_g = \frac{1}{c^2}\frac{\partial^2 \mathbf{H}_g}{\partial t^2}. \tag{3.11}$$

Take now the constitutive equations in the form

$$\mathbf{D}_g = M\mathbf{B}_g + \frac{N}{c^2}\mathbf{E}_g, \tag{3.12}$$

$$\mathbf{H}_g = N\mathbf{B}_g - M\mathbf{E}_g, \tag{3.13}$$

where N, M are constants. In absence of sources, the Maxwell equations become

$$\operatorname{curl}\mathbf{E}_g = -\frac{\partial \mathbf{B}_g}{\partial t}, \quad \operatorname{div}\mathbf{B}_g = 0, \tag{3.14}$$

$$\operatorname{curl}\mathbf{H}_g = \frac{\partial \mathbf{D}_g}{\partial t}, \quad \operatorname{div}\mathbf{D}_g = 0. \tag{3.15}$$

If we express the Maxwell equations through $\mathbf{E}_g$ and $\mathbf{B}_g$, the second pair of equations become

$$\operatorname{curl}\mathbf{H}_g = \frac{\partial \mathbf{D}_g}{\partial t} \Rightarrow N\operatorname{curl}\mathbf{B}_g - M\operatorname{curl}\mathbf{E}_g = M\frac{\partial \mathbf{B}_g}{\partial t} + \frac{N}{c^2}\frac{\partial \mathbf{E}_g}{\partial t}. \tag{3.16}$$

Since $\operatorname{curl}\mathbf{E}_g = -\frac{\partial \mathbf{B}_g}{\partial t}$, we get

$$\operatorname{curl}\mathbf{B}_g = \frac{1}{c^2}\frac{\partial \mathbf{E}_g}{\partial t}. \tag{3.17}$$

The second equation, $\operatorname{div}\mathbf{D}_g = 0$, reduces to $M\operatorname{div}\mathbf{B}_g + \frac{N}{c^2}\operatorname{div}\mathbf{E}_g = 0$. Since $\operatorname{div}\mathbf{B}_g = 0$, we get

$$\operatorname{div}\mathbf{E}_g = 0. \tag{3.18}$$

Thus, using constitutive equations with constant M and N we have Maxwell equations in terms of $\mathbf{B}_g$ and $\mathbf{E}_g$, i.e.

$$\operatorname{curl}\mathbf{E}_g = -\frac{\partial \mathbf{B}_g}{\partial t}, \quad \operatorname{div}\mathbf{B}_g = 0, \tag{3.19}$$

$$\operatorname{curl}\mathbf{B}_g = \frac{1}{c^2}\frac{\partial \mathbf{E}_g}{\partial t}, \quad \operatorname{div}\mathbf{E}_g = 0. \tag{3.20}$$

At this stage, we get the wave equations in the standard way. The time derivative of the first equation yields $\operatorname{curl}\frac{\partial}{\partial t}\mathbf{E}_g = -\frac{\partial^2 \mathbf{B}_g}{\partial t^2} \Rightarrow c^2\operatorname{curl}(\operatorname{curl}\mathbf{B}_g) = -\frac{\partial^2 \mathbf{B}_g}{\partial t^2}$. Since curl(curl $\mathbf{B}_g$) = grad(div $\mathbf{B}_g$) − $\Delta\mathbf{B}_g = -\Delta\mathbf{B}_g$, then

$$\Delta\mathbf{B}_g = \frac{1}{c^2}\frac{\partial^2\mathbf{B}_g}{\partial t^2}. \tag{3.21}$$

By analogy curl $\frac{\partial}{\partial t}\mathbf{B}_g = \frac{1}{c^2}\frac{\partial^2\mathbf{E}_g}{\partial t^2} \Rightarrow -\mathrm{curl}(\mathrm{curl}\,\mathbf{E}_g) = \frac{1}{c^2}\frac{\partial^2\mathbf{E}_g}{\partial t^2}$, and we get the wave equation for $\mathbf{E}_g$,

$$\Delta\mathbf{E}_g = \frac{1}{c^2}\frac{\partial^2\mathbf{E}_g}{\partial t^2}. \tag{3.22}$$

Thus, the gravitoelectromagnetic waves $\mathbf{E}_g$ and $\mathbf{B}_g$ have speed c and do not depend on the constants M and N.

4. Waves and Constitutive Equations for Linear Constitutive Functions

Let us consider the constitutive equations (1.13) and (1.14) as linear functions of the invariants, i.e.

$$M = M_g(I_{g1}, I_{g2}) = a_m I_{g1} + b_m I_{g2}, \tag{4.1}$$

$$N = N_g(I_{g1}, I_{g2}) = c^2\varepsilon_g + a_n I_{g1} + b_n I_{g2}, \tag{4.2}$$

a_m, b_m, a_n, b_n being some constants. From all the Maxwell equations in material media, and in the absence of sources one finds curl $\mathbf{H}_g = \frac{\partial\mathbf{D}_g}{\partial t}$, $\mathrm{curl}(N\mathbf{B}_g - M\mathbf{E}_g) = \frac{\partial}{\partial t}(M\mathbf{B}_g + \frac{N}{c^2}\mathbf{E}_g)$, and $N\,\mathrm{curl}\,\mathbf{B}_g - M\,\mathrm{curl}\,\mathbf{E}_g = M\frac{\partial\mathbf{B}_g}{\partial t} + \frac{N}{c^2}\frac{\partial\mathbf{E}_g}{\partial t}$. Since curl $\mathbf{E}_g = -\frac{\partial\mathbf{B}_g}{\partial t}$, from the last equation one gets

$$\mathrm{curl}\,\mathbf{B}_g = \frac{1}{c^2}\frac{\partial\mathbf{E}_g}{\partial t}. \tag{4.3}$$

The second equation, div $\mathbf{D}_g = 0$, reduces to $\mathrm{div}(M\mathbf{B}_g + \frac{N}{c^2}\mathbf{E}_g) = 0$, or $M\,\mathrm{div}\,\mathbf{B}_g + \frac{N}{c^2}\mathrm{div}\,\mathbf{E}_g = 0$. Since div $\mathbf{B}_g = 0$, one gets

$$\mathrm{div}\,\mathbf{E}_g = 0. \tag{4.4}$$

5. Inverse Problem of Nonlinear Gravitoelectromagnetism

In electrodynamics the direct solution of the Maxwell equations together with the nonlinear constitutive equations is a non-trivial and complicated task even for simple systems [1, 2]. In previous sections we presented some very special cases of the nonlinear functions N and M. Here we formulate the following inverse problem: if we have some particular solution of the gravity-Maxwell equations (1.8) and (1.9), can we then find the exact form of the corresponding nonlinear gravity-constitutive equations (1.13) and (1.14)?

It is natural to consider the case of plane gravitational waves, when the fields have only one space coordinate. We will show that even in this case one can have a non-trivial nonlinearity. Let us choose $\mathbf{E}_g$ and $\mathbf{B}_g$ mutually orthogonal and perpendicular to the direction of motion

$$\mathbf{E}_g = \begin{pmatrix} E \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{B}_g = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}, \tag{5.1}$$

where $E \equiv E(t, y), B \equiv B(t, y)$. Now the invariants (1.15) become

$$I_{g1} = B^2 - \frac{1}{c^2} E^2 \equiv I, \tag{5.2}$$

$$I_{g2} = 0. \tag{5.3}$$

The use of the nonlinear gravity-constitutive equations (1.13) and (1.14) gives for the other fields

$$\mathbf{D}_g = \begin{pmatrix} \frac{1}{c^2} NE \\ 0 \\ MB \end{pmatrix}, \quad \mathbf{H}_g = \begin{pmatrix} -ME \\ 0 \\ NB \end{pmatrix}, \tag{5.4}$$

where $N \equiv N(I), M \equiv M(I)$ are the sought for gravity-constitutive functions. They depend on I only, because of Lorentz invariance (see [1, 2]). Inserting the fields (5.1) and (5.4) into the gravity-Maxwell equations (1.8) and (1.9) without sources gives us three equations (hereafter, a prime with the corresponding subscript denotes the first partial derivative with respect to the variable in the subscript, while dot denotes time derivative)

$$E'_y = \dot{B}, \tag{5.5}$$

$$(NB)'_y = \frac{1}{c^2} (NE)^{\cdot}, \tag{5.6}$$

$$(ME)'_y = (MB)^{\cdot}. \tag{5.7}$$

Now we take into account that the gravity-constitutive functions N, M depend only on the invariant I and present (5.6) and (5.7) as the differential equations for them

$$N'_I \left(B I'_y - \frac{1}{c^2} E \dot{I} \right) + N \left(B'_y - \frac{1}{c^2} \dot{E} \right) = 0, \tag{5.8}$$

$$M'_I \left(E I'_y - B \dot{I} \right) = 0, \tag{5.9}$$

where we have exploited the identities

$$\begin{aligned} N'_y &= N'_I I'_y, \quad M'_y = M'_I I'_y, \\ \dot{N} &= N'_I \dot{I}, \quad \dot{M} = M'_I \dot{I}. \end{aligned} \tag{5.10}$$

The Eq. (5.9) can be immediately solved by

$$M(I) = \begin{cases} M_0 = \text{const.}, & \text{if } E I'_y \neq B \dot{I}, \\ \text{arbitrary}, & \text{if } E I'_y = B \dot{I}. \end{cases} \tag{5.11}$$

The Eq. (5.8) can be solved if

$$\lambda \equiv \frac{(B'_y - \frac{\dot{E}}{c^2})}{(BI'_y - \frac{E\dot{I}}{c^2})}, \tag{5.12}$$

depends only on I, which is a very special case. One then has the differential equation

$$N'_I + \lambda(I)N = 0 \tag{5.13}$$

and its solution is

$$N(I) = N_0 \mathrm{e}^{-\int \lambda(I)\mathrm{d}I}. \tag{5.14}$$

Otherwise, by using the expressions for I'_y and $\dot{I}$ from (5.2), i.e.

$$I'_y = 2BB'_y - \frac{2EE'_y}{c^2}, \quad \dot{I} = 2B\dot{B} - \frac{2E\dot{E}}{c^2}, \tag{5.15}$$

we obtain

$$2N'_I \left(B^2 B'_y + \frac{1}{c^4} E^2 \dot{E} - \frac{2}{c^2} EBE'_y \right) + N \left(B'_y - \frac{1}{c^2}\dot{E} \right) = 0, \tag{5.16}$$

where the sum of terms in brackets is not a function of I, in general.

Usually, in the wave solutions the dependence of fields on frequency ω and wave number k is the same, and therefore we can consider the concrete choice

$$E(t,y) = f(\varepsilon\omega t + ky) \equiv f(X(t,y)), \quad B(t,y) = g(\varepsilon\omega t + ky) \equiv g(X(t,y)), \tag{5.17}$$

where $\varepsilon \equiv \pm 1$, with f and g arbitrary smooth nonvanishing functions. Bearing in mind that

$$E'_y = f'_X X'_y = kf'_X, \quad B'_y = g'_X X'_y = kg'_X,$$
$$\dot{E} = f'_X \dot{X} = \varepsilon\omega f'_X, \quad \dot{B} = g'_X \dot{X} = \varepsilon\omega g'_X,$$

our Eq. (5.5) yields

$$kf'_X = \varepsilon\omega g'_X. \tag{5.18}$$

Therefore,

$$g(X) = \frac{k}{\varepsilon\omega} f(X) + \alpha, \tag{5.19}$$

where α is a constant, so that both E and B can be expressed through one function only, i.e. f, and the invariant I reads eventually as

$$I = \frac{1}{\omega^2}\left(k^2 - \frac{\omega^2}{c^2}\right) f^2 + 2\frac{k}{\varepsilon\omega}\alpha f + \alpha^2. \tag{5.20}$$

The equations for the gravity-constitutive functions take therefore the form

$$N'_I \left[2I\left(k^2 - \frac{\omega^2}{c^2}\right) + 2\frac{\omega^2}{c^2}\alpha^2 \right] + N\left(k^2 - \frac{\omega^2}{c^2}\right) = 0, \tag{5.21}$$

$$M'_I f'_X \left[\frac{2f}{\varepsilon\omega}\left(k^2 - \frac{\omega^2}{c^2}\right) + 2k\alpha \right] \alpha = 0, \tag{5.22}$$

having exploited the identities

$$gI'_y - \frac{f\dot{I}}{c^2} = \left(gf'_y - \frac{f\dot{f}}{c^2}\right)\left[\frac{2f}{\omega^2}\left(k^2 - \frac{\omega^2}{c^2}\right) + 2\frac{k}{\varepsilon\omega}\alpha\right], \tag{5.23}$$

$$gf'_y - \frac{f\dot{f}}{c^2} = \frac{f'_X}{\varepsilon\omega}\left[f\left(k^2 - \frac{\omega^2}{c^2}\right) + k\varepsilon\omega\alpha\right], \tag{5.24}$$

and, after some cancellations,

$$\left[f\left(k^2 - \frac{\omega^2}{c^2}\right) + k\varepsilon\omega\alpha\right]\left[\frac{2f}{\omega^2}\left(k^2 - \frac{\omega^2}{c^2}\right) + 2\frac{k}{\varepsilon\omega}\alpha\right]$$
$$= 2\left(k^2 - \frac{\omega^2}{c^2}\right)I + 2\frac{\omega^2}{c^2}\alpha^2, \tag{5.25}$$

while

$$fI'_y - g\dot{I} = (ff'_y - g\dot{f})\left[\frac{2f}{\omega^2}\left(k^2 - \frac{\omega^2}{c^2}\right) + 2\frac{k}{\varepsilon\omega}\alpha\right], \tag{5.26}$$

$$ff'_y - g\dot{f} = f'_X(kf - \varepsilon\omega g) = -\varepsilon\omega f'_X\alpha. \tag{5.27}$$

The results of our analysis now depend on whether or not α vanishes. Indeed, if $\alpha = 0, M$ is arbitrary and hence we obtain the equation

$$\left(k^2 - \frac{\omega^2}{c^2}\right)(2IN'_I + N) = 0, \tag{5.28}$$

which implies that either the dispersion relation

$$k^2 - \frac{\omega^2}{c^2} = 0 \tag{5.29}$$

holds, with N kept arbitrary, or such a dispersion relation is not fulfilled, while N is found from the differential equation

$$2IN'_I + N = 0, \tag{5.30}$$

which is solved by

$$N(I) = \frac{N_0}{\sqrt{I}}. \tag{5.31}$$

By contrast, if α does not vanish, M equals a constant M_0, while N solves the more complicated Eq. (5.21). At this stage, to be consistent with the dependence of N on I only, we have to require again that the dispersion relation (5.29) should hold, jointly with $N'_I = 0$, which implies the constancy of $N : N = N_0$.

6. Concluding Remarks

We have brought "down to earth" the general program of considering nonlinear constitutive equations for gravitoelectromagnetism, by solving the problem of finding, for a given solution of the gravity-Maxwell equations, the exact form of nonlinear constitutive equations. We look forward to being able to construct other relevant examples, as well as being able to re-express our models in the language of differential forms, which turned out to be very powerful for general relativity [9–11].

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